A O(n⁸)×O(n⁷) Linear Programming Model of the Quadratic Assignment Problem

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Abstract: In this paper, we propose a linear programming (LP) formulation of the Quadratic Assignment Problem (QAP) with $O(n^8)$ variables and $O(n^7)$ constraints, where n is the number of assignments. A small experimentation that was undertaken in order to gain some rough indications about the computational performance of the model is discussed.

Keywords: Quadratic Assignment Problem; Linear Programming; Facilities Layout; Combinatorial Optimization; Computational Complexity.

1 Introduction

The Quadratic Assignment Problem (QAP) is the problem of making exclusive assignments of n indivisible entities to n other indivisible entities in such a way that a total quadratic interaction cost is minimized. The problem can be interpreted from a wide variety of perspectives. The perspective adopted in this paper is that of the generic facilities location/layout context, as in the seminal work of Koopmans and Beckmann [8]. Specifically, there are n facilities (or departments) to be located at n possible sites (or locations). The volume of traffic going from facility i to facility j is denoted f_{ij}.The travel distance from site r to site s is denoted d_{rs}. A quadratic "material handling" cost of h_{irjs} is incurred if facilities i and j are assigned to sites r and s, respectively. In addition, there is a fixed cost (an "operating cost"), o_{ir}, associated with operating facility i at site r. It is assumed (without loss of generality) that the units for "distance", "volume of traffic", and "operating cost" have been chosen so that the h_{irjs}'s and o_{ir}'s are commensurable. The problem is that of finding a perfect matching of the facilities and sites so that the total material handling plus facilities operating costs is minimized.

Let $F := \{1, 2, ..., \eta\}$ and $T := \{1, 2, ..., \varsigma\}$ be the sets of facilities and sites, respectively. Without loss of generality, assume $\eta = \varsigma = n$. For $i \in F$ and $r \in T$, let w_{ir} be a 0/1 binary variable that indicates whether facility i is assigned to (or located at) site r ($w_{ir} = 1$), or not ($w_{ir} = 0$). Then, a classical formulation of the QAP is as follows (see [10]):

Problem 1 (*Problem QAP*)

minimize:

$$v(w) := \sum_{i\in F}\sum_{j\in F}\sum_{r\in T}\sum_{s\in T} h_{irjs} w_{ir} w_{js} + \sum_{i\in F}\sum_{r\in T} o_{ir} w_{ir} \tag{1}$$

subject to:

$$\sum_{i\in F} w_{ir} = 1; \quad r \in T \tag{2}$$

$$\sum_{r\in T} w_{ir} = 1; \quad i \in F \tag{3}$$

$$w_{ir} \in \{0,1\}; \quad i \in F, \quad r \in T \tag{4}$$

$$\textit{where}: h_{irjs} = f_{ij} d_{rs} + f_{ji} d_{sr} \tag{5}$$

Problem QAP was shown to be NP-Hard as far back as the 1970's (see [14]). Moreover, it has been known for some time that the Traveling Salesman Problem (TSP; see [9]) and other NP-Complete combinatorial optimization problems (see [7], [11], or [12]) can be modeled as special cases of the problem. Hence, the thrust of research on the problem has been towards the development of heuristic procedures and "tight" lower bounds (see [1], [3], [10], and [13] for extensive reviews).

In this paper, we present a new linear programming (LP) formulation of the Quadratic Assignment Problem (QAP). The modeling approach is similar to those in [5] and [6]. However, the proposed model is an order of size smaller, having $O(n^8)$ variables and $O(n^7)$ constraints, where n is the number of assignments. A small experimentation that was undertaken in order to gain some rough indications about the computational performance of the model is discussed.

The plan of this paper is as follows. The proposed linear programming formulation is developed in section 2. Computational testing and results are discussed in section 3. Conclusions are discussed in section 4.

2 Development of the Formulation

Our approach consists of modeling the QAP in the framework of the multipartite graph G = (V, A) illustrated in *Figure 1*. Nodes in this graph represent (facility, site) pairs. Arcs are labeled with triples $(a, b, c) \in (F, T, F)$, and represent assignment decisions.

Remark 2 *The association we make between* Graph G *and the QAP is to interpret a positive flow into/out of any node of the graph to mean that the corresponding facility and site pair have been assigned to each other.*

Definition 3

1. The set of all the nodes of *Graph G* that have a given facility index in common is referred to as a *level* of the graph. The i^{th} level $(i \in F)$ is denoted $L_i(G) := \{(u, v) \in V \mid u = i\}$.

2. The set of all the nodes of *Graph G* that have a given site index in common is referred to as a *stage* of the graph. The j^{th} stage $(j \in T)$ is denoted $S_j(G) := \{(u, v) \in V \mid v = j\}$.

3. A path in *Graph G* that simultaneously spans the *levels* and the *stages* of *Graph G* is referred to as a *perfect bipartite matching (p.b.m.) path* of the graph.

4. The set of all the p.b.m. paths of *Graph G* is denoted Ω. That is, $\Omega := \{((i_1, 1, i_2), (i_2, 2, i_3), ..., (i_{n-1}, n-1, i_n)) \in A^{n-1} : i_p \neq i_q \ \forall \ (p, q) \in (T, T \backslash \{p\})\}$.

Figure 1: Illustration of *Graph G*

Remark 4

1. There exists a one-to-one correspondence between perfect matchings of the facilities and sites and p.b.m. paths of *Graph G*;

2. There exists a one-to-one correspondence between the perfect matchings of the facilities and sites and feasible solutions to *Problem QAP*;

3. There exists a one-to-one correspondence between the p.b.m paths of *Graph G* and the feasible solutions of *Problem QAP*.

Assumption 5 *Throughout the rest of this paper, it is assumed (w.l.o.g.) that:*

1. The number of assignments is greater than 5 (i.e., $n > 5$);

2. All vectors of variables are column vectors.

Notation 6 *The following notation will be used throughout the rest of this paper:*

1. The set of real numbers is denoted by $\mathcal{R}$;
2. For two column vectors $\mathbf{a}$ and $\mathbf{b}$, $\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = (\mathbf{a}^T, \mathbf{b}^T)^T$ will be written as "($\mathbf{a}$, $\mathbf{b}$)" (where $(\cdot)^T$denotes the transpose of $(\cdot)$), except for where that causes ambiguity;
3. The i^{th} component of a column vector $\mathbf{a}$ is denoted $\mathbf{a}_i$;
4. The notation "$\mathbf{0}$" denotes a column vector of comfortable size that has every entry equal to 0;
5. The notation "$\mathbf{1}$" denotes a column vector of comfortable size that has every entry equal to 1;
6. The convex hull of $(\cdot)$ is denoted $Conv(\cdot)$;
7. The set of extreme points of $(\cdot)$ is denoted $Ext(\cdot)$;
8. The set of *stages* of *Graph G* from which arcs of *Graph G* originate is denoted R; i.e., $R := T \backslash \{n\}$;
9. $\forall\ (i, j, u, v, k, t) \in F^6$, $\forall\ (p, s) \in R^2 : 1 < p < s$, $z_{i,1,jupvkst}$ denotes a non-negative variable that represents the amount of flow in *Graph G* that propagates from arc $(i, 1, j)$ on, to arc (k, s, t), via arc (u, p, v);
10. $\forall\ (i, j, k, t) \in F^4$, $\forall\ (r, s) \in R^2 : 1 \leq r < s$, y_{irjkst} denotes a non-negative variable that represents the total amount of flow in *Graph G* that propagates from *arc* (i, r, j) on, to *arc* (k, s, t).

2.1 Model Constraints

The constraints of the Integer Programming (IP) version of our proposed model are as follows:

$$\sum_{i \in F} \sum_{j \in F} \sum_{v \in F} \sum_{t \in F} z_{i,1,jj,2,vv,3,t} = 1 \tag{6}$$

$$\sum_{v \in F} z_{i,1,jkstvpu} - \sum_{v \in F} z_{i,1,jkstu,p+1,v} = 0;$$
$$i, j, k, t, u \in F; \quad p, s \in R : 1 < s < p < n - 1 \tag{7}$$

$$\sum_{v \in F} z_{i,1,jvpukst} - \sum_{v \in F} z_{i,1,ju,p+1,vkst} = 0;$$
$$i, j, k, t, u \in F; \quad p, s \in R : 1 < p < s - 1, \ s > 3 \tag{8}$$

$$y_{i,1,jupv} - \sum_{k \in F} \sum_{t \in F} z_{i,1,jupvkst} = 0;$$
$$i, j, u, v \in F; \quad p, s \in R : 1 < p < s \tag{9}$$

$$y_{i,1,jkst} - \sum_{u \in F} \sum_{v \in F} z_{i,1,jupvkst} = 0;$$
$$i,j,k,t \in F; \quad p,s \in R : 1 < p < s \tag{10}$$

$$y_{i,1,jkst} \;\; - \sum_{p \in R: 1<p<s} \sum_{v \in F} z_{i,1,jupvkst} - \sum_{p \in R: p>s} \sum_{v \in F} z_{i,1,jkstvpu} = 0;$$
$$i,j,k,t,u \in F; \quad s \in R \backslash \{1\} \tag{11}$$

$$y_{upvkst} - \sum_{i \in F} \sum_{j \in F} z_{i,1,jupvkst} = 0;$$
$$u,v,k,t \in F; \quad p,s \in R, \quad 1 < p < s \tag{12}$$

$$\sum_{(k,t) \in F^2} y_{irjkrt} \;\; + \sum_{\substack{(k,t) \in F^2: \\ k \neq j;\; (k,r+1,t) \in A}} y_{irjk,r+1,t} = 0; \quad i,j \in F; \quad r \in R \tag{13}$$

$$\sum_{s \in R:\; s \geq r} \sum_{k \in F} \sum_{t \in F} y_{irikst} + \sum_{s \in R:\; s \leq r} \sum_{k \in F} \sum_{t \in F} y_{kstjrj} \sum_{s \in R: s \geq r+1} \sum_{k \in F} y_{irjksi} \;\; +$$
$$+ \sum_{s \in R: s \geq r+1} \sum_{k \in F} y_{irjisk} + \sum_{s \in R: s \geq r+1} \sum_{k \in F} y_{irjksj} + \sum_{s \in R: s \geq r+2} \sum_{k \in F} y_{irjjsk} = 0;$$

$$i,j \in F; \quad r \in R \tag{14}$$

$$y_{irjkst} \in \{0,1\}; \quad i,j,k,\; t \in F \quad r,s \in R : r < s; \tag{15}$$

$$z_{i,1,jupvkst} \in \{0,1\}; i,\; j,u,\; v,\; k,\; t \in F; \quad p,s \in R : 1 < p < s \tag{16}$$

Constraint (6) initiates the propagation of one unit of flow from *stage 1* to *stage 3* of *Graph G*. Constraints (7) stipulate that the total amount of flow from *arc* $(i,1,j)$ that propagates through *arc* (k,s,t) and enters *node* $(u,p+1)$ is equal to the amount of flow from *arc* $(i,1,j)$ that propagates through *arc* (k,s,t) and leaves *node* $(u,p+1)$. Constraints (8) stipulate that the total amount of flow from *arc* $(i,1,j)$ that enters *node* $(u,p+1)$ to propagate on to *arc* (k,s,t) is equal to the amount of flow from *arc* $(i,1,j)$ that leaves *node* $(u,p+1)$ to propagate on to *arc* (k,s,t). Constraints (9) and (10) ensure that the propagation of the flow from a given arc at *stage* 1 of *Graph G* onto a given arc at another given stage of the graph is consistently accounted across all the other stages of the graph. Constraints (12) stipulate that the total amount of flow that propagates from *arc*

(u,p,v) onto *arc* (k,s,t) is equal to the total of the flows from arcs at *stage* 1 that propagate onto *arc* (k,s,t) via *arc* (u,p,v). Constraints (11) stipulate, essentially, that the total flow on any given arc of *Graph G* must propagate on to every *level* of the graph, or be part of a flow propagation that spans the *levels* of the graph. Constraints (13) ensure that the initial flow propagation from any given arc occurs in an "unbroken" fashion. Finally, constraints (14) stipulate (in light of the other constraints) that flow from *arc* (i,r,j) of *Graph G* cannot propagate back onto neither *level* i nor *level* j of the graph.

Theorem 7 *The number of variables in the system (6)-(14) is* $O(n^8)$*; The number of constraints in the system (6)-(14) is* $O(n^7)$.

Proof. Trivial. ■

Definition 8

1. We refer to the set of points in the space of the y- and z-variables that satisfy the system (6)-(16) as the "IP Polytope," and denote it by Q_I; i.e., $Q_I := \{(y,z) \in \mathcal{R}^{\xi} : (y,z)\ satisfies$ (6)-(16)$\}$, where ξ is the number of variables in the system (6)-(16).

2. We refer to the linear programming relaxation of Q_I as the "LP Polytope," and denote it by Q_L; i.e., $Q_L := \{(y,z) \in \mathcal{R}^{\xi} : (y,z)$ satisfies (6)-(14), and $\mathbf{0} \leq (y,\ z) \leq \mathbf{1}\}$, where ξ is the number of variables in the system (6)-(14).

3. The set of points in the space of the w-variables that satisfy the system (2)-(4) is referred to as the "Assignment Polytope," and is denoted by W_I; i.e., $W_I := \{w \in \mathcal{R}^{\nu} : w$ satisfies (2)-(4)$\}$, where ν is the number of variables in the system (2)-(4);

4. The linear programming relaxation of W_I is denoted by W_L; i.e., $W_L := \{w \in \mathcal{R}^{\nu} : w\ satisfies$ (2)-(3), and $\mathbf{0} \leq w \leq \mathbf{1}\}$, where ν is the number of variables in the system (2)-(4).

2.1.1 Structure of the IP Polytope

Theorem 9 $(y,z) \in Q_I \iff \exists$ *exactly one set of facility indices,* $\{i_r \in F,\ r = 1,\ldots,n\}$*, such that:*

$$y_{arbcsd} = \begin{cases} 1 & \textit{for } r,s \in R : r < s, \textit{ and} \\ & (a,b,c,d) = (i_r, i_{r+1}, i_s, i_{s+1}) \\ & \\ 0 & \textit{otherwise} \end{cases} \tag{17}$$

and

$$z_{a,1,bcrdesf} = \begin{cases} 1 & \textit{for } r,s \in R : 1 < r < s, \textit{ and} \\ & (a,b,c,d,e,f) = (i_1, i_2, i_r, i_{r+1}, i_s, i_{s+1}) \\ & \\ 0 & \textit{otherwise} \end{cases} \tag{18}$$

Proof. *a)* $\Longrightarrow$: Let $(y,z) \in Q_I$. Then, given (15), and (16):

Constraint (6) $\Longrightarrow \exists$ a unique set of facility indices, $\{i_r \in F,\ r = 1, \ldots, 4\}$, such that:

$$z_{i_1,1,i_2i_2,2,i_3i_3,3,i_4} = 1 \tag{19}$$

Condition (18) follows directly from the combination of (19), (7), and (8).

Condition (17) follows from the combination of Condition (18) with constraints (9)-(10) and (12).

b) $\Longleftarrow$: Trivial. ■

Theorem 10 *There exists a one-to-one correspondence between the feasible solutions to the system (6)-(16), and the perfect matchings of the facilities and sites.*

Proof. Combining *Theorem* 9 with constraints (14), (15), and (16), we must have:

$$(y,z) \in Q_I \Longleftrightarrow \exists (i_1, i_2, ..., i_n) \in F^n\colon \begin{cases} i) & (17) \text{ and } (18) \text{ are satisfied for } (y,z), \text{ and} \\ ii) & i_p \neq i_q \ \forall (p,q) \in (T, T\backslash\{p\}) \end{cases} \tag{20}$$

The theorem follows directly from the combination of (20), the definition of the y- and z-variables (see Notation 6.9 and 6.10), Definition 3.4, and Remark 4.1. ■

Corollary 11 *Q_I is isomorphic to Ω and to W_I, respectively.*

Definition 12 *We refer to the perfect matching of the facilities and sites correponding to $(y,z) \in Q_I$ as the "assignment corresponding to (y,z)," and denote it by the ordered set $\mathcal{M}(y,z) := \langle i_1, i_2, \cdots, i_n \rangle$, where i_q is the index of the facility assigned to site q in the matching.*

2.1.2 Structure of the LP Polytope

Lemma 13 (Flow propagation lemma 1) *Let $(y,z) \in Q_L$. The following holds true:*

$$\forall (i_1, i_2, i_3, i_4) \in F^4, \quad y_{i_1,1,i_2,i_3,3,i_4} > 0 \Longleftrightarrow z_{i_1,1,i_2,i_2,2,i_3,i_3,3,i_4} > 0. \tag{21}$$

Proof. Using constraints (9) and (13), constraints (10) for $p = 2$ and $s = 3$ can be written as:

$$y_{i_1,1,i_2i_3,3,i_4} - z_{i_1,1,i_2i_2,2,i_3i_3,3,i_4} = 0 \ \ \forall (i_1, i_2, i_3, i_4) \in F^4 \tag{22}$$

The lemma follows directly from (22). ■

Lemma 14 (Flow propagation lemma 2) *Let $(y,z) \in Q_L$. Then, we must have that:*

$$\forall r \in R : r \geq 4, \quad \forall (i_1, i_2, i_3, i_4, i_r, i_{r+1}) \in F^6,$$

$$z_{i_1,1,i_2,i_3,3,i_4,i_r,r,i_{r+1}} > 0 \Longrightarrow \begin{cases} i) \ z_{i_1,1,i_2,i_2,2,i_3,i_3,3,i_4} > 0; \\ ii) \ z_{i_1,1,i_2,i_2,2,i_3,i_r,r,i_{r+1}} > 0. \end{cases} \tag{23}$$

Proof.

a) Condition i. Using constraints (9),

$$z_{i_1,1,i_2,i_3,3,i_4,i_r,r,i_{r+1}} > 0 \Longrightarrow \ y_{i_1,1,i_2,i_3,3,i_4} > 0 \ \ \forall r \in R : r \geq 4, \ \ \forall (i_1, i_2, i_3, i_4, i_r, i_{r+1}) \in F^6 \tag{24}$$

From Lemma 13,

$$y_{i_1,1,i_2,i_3,3,i_4} > 0 \Longrightarrow z_{i_1,1,i_2,i_2,2,i_3,i_3,3,i_4} > 0 \ \ \forall (i_1, i_2, i_3, i_4) \in F^4 \tag{25}$$

Condition i) follows directly from (25).

b) Condition ii. Using (9), (10), (12), and (13), constraints (8) for $p = 2$ and $u = i_3$, can be written as:

$$z_{i_1,1,i_2,i_2,2,i_3,i_s,s,i_{s+1}} - \sum_{v \in F} z_{i_1,1,i_2,i_3,3,v,i_s,s,i_{s+1}} = 0 \ \forall (i_1, i_2, i_3, i_s, i_{s+1}) \in F^5; \ \ \forall s \in R : s \geq 4 \tag{26}$$

Hence, in particular, we must have:

$$z_{i_1,1,i_2,i_2,2,i_3,i_r,r,i_{r+1}} - z_{i_1,1,i_2,i_3,3,i_4,i_r,r,i_{r+1}} \geq 0 \ \ \forall (i_1, i_2, i_3, i_4, i_r, i_{r+1}) \in F^6; \ \ \forall \ r \in R : r \geq 4 \tag{27}$$

Condition ii follows directly from (27). ■

Lemma 15 (Flow propagation lemma 3) *The following holds true for all* $(y, z) \in Q_L$*:*

$$\forall r \in R : 2 \leq r \leq n-3, \ \forall (i_1, i_2, i_r, i_{r+1}, i_{r+2}, i_{r+3}) \in F^6,$$

$$z_{i_1,1,i_2,i_r,r,i_{r+1},i_{r+2},r+2,i_{r+3}} > 0 \Longrightarrow \begin{cases} i) \ z_{i_1,1,i_2,i_{r+1},r+1,i_{r+2},i_{r+2},r+2,i_{r+3}} > 0; \\ \\ ii) \ z_{i_1,1,i_2,i_r,r,i_{r+1},i_{r+1},r+1,i_{r+2}} > 0. \end{cases} \tag{28}$$

Proof. *a) Condition i.* Using (9), (10), (12), and (13), constraints (8) for $p = r$, $s = r + 2$, and $u = i_{r+1}$, can be written as:

$$\sum_{v \in F} z_{i_1,1,i_2,v,r,i_{r+1},i_{r+2},r+2,i_{r+3}} - z_{i_1,1,i_2,i_{r+1},r+1,i_{r+2},i_{r+2},r+2,i_{r+3}} = 0$$
$$\forall r \in R : 2 \leq r \leq n-3, \ \forall (i_1, i_2, i_{r+1}, i_{r+2}, i_{r+3}) \in F^5 \tag{29}$$

Hence, in particular, we must have:

$$z_{i_1,1,i_2,i_r,r,i_{r+1},i_{r+2},r+2,i_{r+3}} - z_{i_1,1,i_2,i_{r+1},r+1,i_{r+2},i_{r+2},r+2,i_{r+3}} \leq 0$$
$$\forall r \in R : 2 \leq r \leq n-3, \ \forall (i_1, i_2, i_r, i_{r+1}, i_{r+2}, i_{r+3}) \in F^6 \tag{30}$$

Condition i) follows directly from (30).

b) Condition ii. Using constraints (9), (10), (12), and (13), constraints (7) for $p = r + 1$, $s = r$, and $u = i_{r+2}$, can be written as:

$$z_{i_1,1,i_2,i_r,r,i_{r+1},i_{r+1},r+1,i_{r+2}} - \sum_{v \in F} z_{i_1,1,i_2,i_r,r,i_{r+1},i_{r+2},r+2,v} = 0$$
$$\forall r \in R : 2 \leq r \leq n-3, \ \forall (i_1, i_2, i_{r+1}, i_{r+2}, i_{r+3}) \in F^5 \tag{31}$$

Hence, in particular, we must have:

$$z_{i_1,1,i_2,i_r,r,i_{r+1},i_{r+1},r+1,i_{r+2}} - z_{i_1,1,i_2,i_r,r,i_{r+1},i_{r+2},r+2,i_{r+3}} \geq 0$$
$$\forall r \in R : 2 \leq r \leq n-3,\ \forall (i_1, i_2, i_r, i_{r+1}, i_{r+2}, i_{r+3}) \in F^6 \tag{32}$$

Condition ii) follows directly from (32). ∎

Notation 16 *For $(y, z) \in Q_L$:*

1. The sub-graph of G induced by the positive components of (y, z) is denoted as:

$$H(y,z) := (P(y,z)), E(y,z)), \tag{33}$$

   where:

$$P(y,z) := \left\{ (i,1) \in V : \sum_{j \in F} \sum_{t \in F} y_{i,1,jj,2,t} \quad > \quad 0 \right\} \cup$$
$$\left\{ (i,r) \in V : \sum_{a \in F} \sum_{b \in F} \sum_{j \in F:(i,r,j) \in A} y_{a,1,birj} \ + \right.$$
$$\left. + \ \sum_{a \in F} \sum_{b \in F} \sum_{j \in F:(j,r-1,i) \in A} y_{a,1,bj,r-1,i} \quad > \quad 0 \right\} \tag{34}$$

$$E(y,z) := \left\{ (i,1,j) \in A : \sum_{t \in F} y_{i,1,jj,2,t} \quad > \quad 0 \right\} \cup$$
$$\left\{ (i,r,j) \in A : \sum_{a \in F} \sum_{b \in F} \sum_{j \in F:(i,r,j) \in A} y_{a,1,birj} \quad > \quad 0 \right\}. \tag{35}$$

2. The set of arcs of $H(y,z)$ originating at stage r of $H(y,z)$ is denoted $\Gamma_r(y,z)$;

3. The number of arcs originating at stage r of Graph $H(y,z)$ is denoted $\gamma_r(y,z) = |\Gamma_r(y,z)|$. For simplicity $\gamma_r(y,z)$ will be henceforth written as γ_r (unless that causes ambiguity);

4. The index set associated with $\Gamma_r(y,z)$ is denoted $\Lambda_r(y,z) := \{1, 2, \ldots, \gamma_r\}$. For simplicity $\Lambda_r(y,z)$ will be henceforth written as Λ_r;

5. The ν^{th} arc in $\Gamma_r(y,z)$ is denoted as $a_{r,\nu}(y,z)$. For simplicity $a_{r,\nu}(y,z)$ will be henceforth written as $a_{r,\nu}$;

6. The tail of $a_{r,\nu}$ is labeled $i_{r,\nu}(y,z)$; the head of $a_{r,\nu}(y,z)$ is labeled $j_{r,\nu}(y,z)$. For simplicity, $i_{r,\nu}(y,z)$ will be henceforth written as $i_{r,\nu}$, and $j_{r,\nu}(y,z)$, as $j_{r,\nu}$;

7. Where that causes no confusion (and where that is convenient), for $r \in R$ with $r \geq 2$, and $(\alpha,\ \rho) \in (\Lambda_1, \Lambda_r)$ "$y_{i_{1,\alpha},1,j_{1,\alpha},i_{r,\rho},r,j_{r,\rho}}$" will be henceforth written as "$y_{(1,\alpha)(r,\rho)}$." Similarly, for $(r,s) \in R^2$ with $1 < r < s$ and $(\alpha, \rho, \sigma) \in (\Lambda_1, \Lambda_r, \Lambda_s)$, "$z_{i_{1,\alpha},1,j_{1,\alpha},i_{r,\rho},r,j_{r,\rho},i_{s,\sigma},s,j_{s,\sigma}}$" will be henceforth written as "$z_{(1,\alpha)(r,\rho)(s,\sigma)}$."

Lemma 17 (Flow conservation lemma 1) *Let $(y,z) \in Q_L$. The following holds true: $\forall \alpha \in \Lambda_1$, $\forall (p,q,r,s) \in R^4 : 1 < p < q;\ 1 < p < r < s$,*

$$\sum_{\nu_p \in \Lambda_p} \sum_{\nu_q \in \Lambda_q} z_{(1,\alpha)(p,\nu_p)(q,\nu_q)} = \sum_{\nu_r \in \Lambda_r} \sum_{\nu_s \in \Lambda_s} z_{(1,\alpha)(r,\nu_r)(s,\nu_s)}$$

Proof. $\forall \alpha \in \Lambda_1$, $\forall (p,q,r,s) \in R^4 : 1 < p < q;\ 1 < p < r < s$,

$$\begin{aligned} \sum_{\nu_p \in \Lambda_p} \sum_{\nu_q \in \Lambda_q} z_{(1,\alpha)(p,\nu_p)(q,\nu_q)} &= \sum_{\nu_p \in \Lambda_p} y_{(1,\alpha)(p,\nu_p)} \quad \text{(Using (9))} \\ &= \sum_{\nu_p \in \Lambda_p} \sum_{\nu_r \in \Lambda_r} z_{(1,\alpha)(p,\nu_p)(r,\nu_r)} \quad \text{(Using (9))} \\ &= \sum_{\nu_r \in \Lambda_r} y_{(1,\alpha)(r,\nu_r)} \quad \text{(Using (10))} \\ &= \sum_{\nu_r \in \Lambda_r} \sum_{\nu_s \in \Lambda_s} z_{(1,\alpha)(r,\nu_r)(s,\nu_s)} \quad \text{(Using (9))} \end{aligned}$$

■

Definition 18 ("*Paths in (y,z)*") *Let $(y,z) \in Q_L$. $\forall\ (r,s) \in R^2 : s \geq \max\{3, r{+}1\}$, $\forall (\nu_1, \nu_r, \nu_s) \in (\Lambda_1, \Lambda_r, \Lambda_s)$, a set of arcs of $H(y,z)$,*

$$\big\{(a_{r,\nu_r}, \ldots, a_{s,\nu_s}) \in (E(y,z))^{s-r+1} : z_{(1,\nu_1)(p,\nu_p)(q,\nu_q)} > 0 \ \ \forall (p,q) \in R^2 : \\ \max\{2,r\} \leq p < q < s-1; \quad i_{p,\nu_p} = j_{p-1,\nu_{p-1}} \ \forall p \in (R \cap [r+1,s])\big\}$$

is referred to as a "path in (y,z) from (r,ν_r) to (s,ν_s)."

Notation 19 *Let $(y,z) \in Q_L$. $\forall\ (r,s) \in R^2 : s \geq \max\{3, r+1\}$, $\forall\ (\rho,\sigma) \in (\Lambda_r, \Lambda_s)$,*

1. The set of all *paths in (y,z) from (r,ρ) to (s,σ)* is denoted $U_{(r,\rho)(s,\sigma)}(y,z)$;

2. The index set associated with $U_{(r,\rho)(s,\sigma)}(y,z)$ is denoted $\Phi_{(r,\rho)(s,\sigma)}(y,z) :=$ $\{1, 2, \ldots, \varphi_{(r,\rho)(s,\sigma)}(y,z)\}$, where $\varphi_{(r,\rho)(s,\sigma)}(y,z) := \left|U_{(r,\rho)(s,\sigma)}(y,z)\right|$;

3. The k^{th} element of $U_{(r,\rho)(s,\sigma)}(y,z)$ ($k \in \Phi_{(r,\rho)(s,\sigma)}(y,z)$) is denoted $\mathcal{L}_{(r,\rho),(s,\sigma),k}(y,z)$.

Theorem 20 (Distinct *paths in (y,z)*) *Let $(y,z) \in Q_L$. Then, $\forall\ (r,s) \in R^2 : s \geq \max\{3, r{+}1\}$, $\forall (\alpha_1, \alpha_2) \in \Lambda_r^2$, $\forall (\beta_1, \beta_2) \in \Lambda_s^2 : U_{(r,\alpha_1)(s,\beta_1)}(y,z) \neq \emptyset; U_{(r,\alpha_2)(s,\beta_2)}(y,z) \neq \emptyset$, $\forall\ k \in \Phi_{(r,\alpha_1)(s,\beta_1)}(y,z)$, $\forall\, t \in \Phi_{(r,\alpha_2)(s,\beta_2)}(y,z)$, $\mathcal{L}_{(r,\alpha_1),(s,\beta_1),k}(y,z) \neq \mathcal{L}_{(r,\alpha_2),(s,\beta_2),t}(y,z) \iff \exists\, \{g \in R : r \leq g \leq s;\ \ (\gamma_1, \gamma_2) \in (\Lambda_g, \Lambda_g \backslash \{\gamma_1\})\} \ni \left\{a_{g,\gamma_1} \in \mathcal{L}_{(r,\alpha_1),(s,\beta_1),k}(y,z);\ a_{g,\gamma_2} \in \ \mathcal{L}_{(r,\alpha_2),(s,\beta_2),t}(y,z)\right\}$.*

Proof. The theorem follows directly from the combination of constraints (9), (10), (12), and (13), and Definition 18. ■

Theorem 21 (Path structure theorem 1) *Let* $(y,z) \in Q_L$. *The following holds true:* $\forall (r,s) \in R^2$ *with* $s \geq \max\{3, r+1\}$, $\forall (\rho,\sigma) \in (\Lambda_r, \Lambda_s)$, $y_{(r,\rho)(s,\sigma)} > 0 \Longleftrightarrow U_{(r,\rho)(s,\sigma)}(y,z) \neq \emptyset$.

Proof. First, note that it follows directly from the combination of Lemmas 13, 14, and 15, that the theorem holds true for all $(r,s) \in R^2$ with $s \in \{r+1,\ r+2\}$, and all $(\nu_r,\nu_s) \in (\Lambda_r,\Lambda_s)$.

a) $\Longrightarrow$: Assume there exists an integer $\omega \geq 2$ such that the theorem holds true for all $(p,t) \in R^2$ with $t = p+\omega$, and all $(\nu_p,\nu_t) \in (\Lambda_p,\Lambda_t)$. We will show that the theorem must hold for all $(p,u) \in R^2$ with $u = t+1 = p+\omega+1$, and all $(\nu_p,\nu_u) \in (\Lambda_r,\Lambda_u)$.

Let $(p,u) \in R^2$ with $u = p+\omega+1$, and $(\nu_p,\nu_u) \in (\Lambda_p,\Lambda_u)$ be such that:

$$y_{(p,\nu_p)(u,\nu_u)} > 0. \tag{36}$$

Define:

$$B_{(p,\nu_p)(u,\nu_u)}(y,z) := \{\alpha \in \Lambda_1 : z_{(1,\alpha)(p,\nu_p)(u,\nu_u)} > 0\}. \tag{37}$$

Then, (9), (10) and (36)$\Longrightarrow$

$$\begin{cases} i) \ \ B_{(p,\nu_p)(u,\nu_u)}(y,z) \neq \varnothing, \quad \text{with} \\ ii) \ \ y_{(p,\nu_p)(u,\nu_u)} = \displaystyle\sum_{\alpha \in B_{(p,\nu_p)(u,\nu_u)}(y,z)} z_{(1,\alpha)(p,\nu_p)(u,\nu_u)} \end{cases} \tag{38}$$

Condition (38), and constraints (8) and (13) $\Longrightarrow$

$$\forall\ \alpha \in B_{(p,\nu_p)(u,\nu_u)}(y,z), \quad \exists\ C_{\alpha,(p,\nu_p)(u,\nu_u)}(y,z) \subseteq \Lambda_{p+1} \quad \ni \ :$$

$$\begin{cases} i) \ \ i_{p+1,\beta} = j_{p,\nu_p} \quad \forall\ \beta \in C_{\alpha,(p,\nu_p)(u,\nu_u)}(y,z) \\ ii) \ \ z_{(1,\alpha)(p+1,\beta)(u,\nu_u)} > 0 \quad \forall\ \beta \in C_{\alpha,(p,\nu_p)(u,\nu_u)}(y,z); \text{ and} \\ iii) \ \ z_{(1,\alpha)(p,\nu_p)(u,\nu_u)} \leq \displaystyle\sum_{\beta \in C_{\alpha,(p,\nu_p)(u,\nu_u)}(y,z)} z_{(1,\alpha)(p+1,\beta)(u,\nu_u)}. \end{cases} \tag{39}$$

By assumption (since $u = (p+1)+\omega$), condition (39.*ii*) $\Longrightarrow$

$$U_{(p+1,\beta)(u,\nu_u)}(y,z) \neq \emptyset \ \ \forall \beta \in C_{\alpha,(p,\nu_p)(u,\nu_u)}(y,z) \tag{40}$$

Also, it follows from the combination of condition (36), constraints (7), and constraints (11), that:

$$\begin{aligned} &\forall\ \beta \in C_{\alpha,(p,\nu_p)(u,\nu_u)}(y,z),\ \exists\ \left\{\Upsilon_{(p,\nu_p)(p+1,\beta)(u,\nu_u)}(y,z) \subseteq \Phi_{(p+1,\beta)(u,\nu_u)}(y,z)\right\} \quad \ni \ : \\ &\big\{(z)_{(1,\alpha)(p,\nu_p)(q,\nu_{q,\iota})} > 0 \quad \forall \iota \in \Upsilon_{(p,\nu_p)(p+1,\beta)(u,\nu_u)}(y,z), \\ &\forall q \in (R \cap [p+1,u]),\ \text{and}\ \forall\ \nu_{q,\iota} \in \Lambda_q : a_{p,\nu_{p,\iota}} \in\ \mathcal{L}_{(p+1,\beta)(u,\nu_u),\iota}(y,z)\big\} \end{aligned} \tag{41}$$

Hence, $\forall\ \beta \in C_{\alpha,(p,\nu_p)(u,\nu_u)}(y,z)$ and $\forall\ \iota \in \Upsilon_{(p,\nu_p)(p+1,\beta)(u,\nu_u)}(y,z)$,

$$\overline{L} := \left(\mathcal{L}_{(p+1,\beta)(u,\nu_u),\iota}(y,z) \quad \cup \quad \{a_{p,\nu_p}\}\right)$$

is a *path in (y,z) from* (p,ν_p) *to* (u,ν_u). Hence, we have that $U_{(p,\nu_p)(u,\nu_u)}(y,z) \neq \emptyset$.

b) $\Longleftarrow$: Follows directly from Definition (18) and constraints (12). ■

Corollary 22 *Let $(y,z) \in Q_L$. The following hold true:*

i) $\forall\ s \in R\backslash\{1\}$, $\forall(\alpha,\sigma) \in (\Lambda_1,\Lambda_s)$, $y_{(1,\alpha)(s,\sigma)} > 0 \Longleftrightarrow U_{(1,\alpha)(s,\sigma)}(y,z) \neq \emptyset$.

ii) $\forall\ (r,s) \in (R\backslash\{1\})^2$ *with* $s \geq \max\{3, r+1\}$, $\forall(\alpha,\rho,\sigma) \in (\Lambda_1,\Lambda_r,\Lambda_s)$,

$$z_{(1,\alpha)(r,\rho)(s,\sigma)} > 0 \Longleftrightarrow \begin{cases} ii.1) \quad U_{(1,\alpha)(s,\sigma)}(y,z) \neq \emptyset, \quad \textit{and} \\ \\ ii.2) \quad \exists\ \kappa \in \Phi_{(1,\alpha)(s,\sigma)}(y,z)\ \ \ni\ a_{r,\rho} \in \mathcal{L}_{(1,\alpha),(s,\sigma),\kappa}(y,z). \end{cases}$$

Definition 23 (*"p.b.m. path in (y,z)"*) *Let $(y,z) \in Q_L$. $\forall(\nu_1,\nu_{n-1}) \in (\Lambda_1,\Lambda_{n-1})$, a* path in (y,z) from $(1,\nu_1)$ to $(n-1,\nu_{n-1})$ *is referred to as a "perfect bipartite matching (p.b.m.) path in* (y,z) (from $(1,\nu_1)$ to (n,ν_{n-1}))*."*

Notation 24 *Let $(y,z) \in Q_L$. For all $(\alpha,\beta) \in (\Lambda_1,\Lambda_{n-1})$,*

1. The set of all *paths in (y,z) from $(1,\alpha)$ to $(n-1,\beta)$* is denoted as $\Pi_{\alpha\beta}(y,z)$;
2. The index set associated with $\Pi_{\alpha\beta}(y,z)$ is denoted $\Psi_{\alpha\beta}(y,z) := \{1,\ 2,\ \ldots,\ \pi_{\alpha\beta}(y,z)\}$, where $\pi_{\alpha\beta}(y,z) := |\Pi_{\alpha\beta}(y,z)|$;
3. The k^{th} element of $\Pi_{\alpha\beta}(y,z)$ is denoted $\mathcal{P}_{\alpha\beta k}(y,z)$.

Remark 25 *Let $(y,z) \in Q_L$. $\forall\ (\alpha,\beta) \in (\Lambda_1,\Lambda_{n-1})$,*

1. $\Pi_{\alpha\beta}(y,z) = U_{(1,\alpha)(n-1,\beta)}(y,z)$;
2. $\Psi_{\alpha\beta}(y,z) = \Phi_{(1,\alpha),(n-1,\beta)}(y,z)$;
3. $\pi_{\alpha\beta}(y,z) = \varphi_{(1,\alpha)(n-1,\beta)}(y,z)$;
4. We assume (w.l.o.g.) that: $\mathcal{P}_{\alpha\beta k}(y,z) = \mathcal{L}_{(1,\alpha),(n-1,\beta),k}(y,z)\ \ \forall\ k \in \varphi_{\alpha\beta}(y,z)$.

Theorem 26 (Equivalence of p.b.m. paths and 2-matching solutions) *For $(y,z) \in Q_L$, every p.b.m. path in (y,z) corresponds to exactly one perfect matching of the facilities and sites.*

Proof. Definition 18, constraints (14), and Defintions 3.3 -3.4 imply that every *p.b.m path in* (y,z) corresponds to exactly one *p.b.m path* of *Graph G.* The theorem follows directly from the combination of this Remark 4.3. ■

Theorem 27 ("Convex independence" of p.b.m. paths) *For $(y,z) \in Q_L$, a given p.b.m. path in (y,z) cannot be represented as a convex combination of other p.b.m. paths in (y,z).*

Proof. Theorem 26 implies that every *p.b.m. path in (y,z)* corresponds to an extreme point of $Conv(W_L) = Conv(W_I)$. The theorem follows directly from this. ■

Theorem 28 (Path structure theorem 2) *Let $(y,z) \in Q_L$. The following holds true: $\forall\ r \in R$, $\forall\ \rho \in \Lambda_r$, $\exists\ \{(\alpha,\beta) \in (\Lambda_1,\Lambda_{n-1});\ \iota \in \Psi_{\alpha\beta}(y,z)\}\ \ \ni\ a_{r,\rho} \in \mathcal{P}_{\alpha\beta\iota}(y,z)$.*

Proof.

Case 1: r = 1. From (34) and (35):

$$a_{r,\rho} \in E(y,z) \Longrightarrow \exists\ \alpha\ \in\ \Lambda_2 \quad \ni \quad y_{(r,\rho)(2,\alpha)} > 0. \tag{42}$$

Condition (42) and constraints (9) $\Longrightarrow$

$$\exists\ \gamma \in \Lambda_{n-1} \quad \ni \quad z_{(r,\rho)(2,\alpha)(n-1,\gamma)} > 0. \tag{43}$$

Condition (43) and constraints (9) $\Longrightarrow$

$$\exists\ \gamma \in \Lambda_{n-1} \quad \ni \quad y_{(r,\rho)(n-1,\gamma)} > 0. \tag{44}$$

The theorem follows from the combination of (44) with Theorem 21.

Case 2: r = n-1. From (34) and (35):

$$a_{r,\rho} \in E(y,z) \Longrightarrow \exists\ \alpha\ \in\ \Lambda_1 \quad \ni \quad y_{(1,\alpha)(r,\rho)} > 0. \tag{45}$$

The theorem follows from the combination of (45) with Theorem 21.

Case 3: 1 < r < n-1. From (34) and (35):

$$a_{r,\rho} \in E(y,z) \Longrightarrow \exists\ \alpha\ \in\ \Lambda_1 \quad \ni \quad y_{(1,\alpha)(r,\rho)} > 0. \tag{46}$$

Condition (46) and constraints (9) $\Longrightarrow$

$$\exists \gamma \in \Lambda_{n-1} \quad \ni \quad z_{(1,\alpha)(r,\rho)(n-1,\gamma)} > 0. \tag{47}$$

The theorem follows from the combination of (47) with Corollary 22*ii*. ∎

Corollary 29 *Let $(y,z) \in Q_L$. The following hold true:*
i) $\forall \alpha \in \Lambda_1$, $\forall r \in R\backslash\{1\}$, $\forall \rho \in \Lambda_r$,

$$y_{(1,\alpha)(r,\rho)} > 0 \Longleftrightarrow \exists\ (\beta \in \Lambda_{n-1};\ \iota \in \Psi_{\alpha\beta}(y,z)) \ni \quad a_{r,\rho} \in \mathcal{P}_{\alpha\beta\iota}(y,z); \tag{48}$$

ii) $\forall (r,s) \in R^2 : 1 < r < s$, $\forall (\alpha,\rho,\sigma) \in (\Lambda_1, \Lambda_r, \Lambda_s)$,

$$z_{(1,\alpha)(r,\rho)(s,\sigma)} > 0 \Longleftrightarrow \exists\ (\beta \in \Lambda_{n-1};\ \iota \in \Psi_{\alpha\beta}(y,z)) \ni \quad (a_{r,\rho}, a_{s,\sigma}) \in \mathcal{P}^2_{\alpha\beta\iota}(y,z). \tag{49}$$

Lemma 30 (Flow conservation lemma 2) *Let $(y,z) \in Q_L$. The following hold true:*

$$i)\ y_{(1,\alpha)(r,\nu_r)} = \sum_{\nu_p \in \Lambda_p} \sum_{\beta \in \Lambda_{n-1}} \sum_{\substack{\iota \in \Psi_{\alpha\beta}(y,z): \\ a_{p,\nu_p} \in \mathcal{P}_{\alpha\beta\iota}(y,z)}} z_{(1,\alpha)(p,\nu_p)(r,\nu_r)}$$

$$\forall \alpha \in \Lambda_1,\ \forall (p,r) \in R^2 : 1 < p < r,\ \forall\ \nu_r \in \Lambda_r$$

$$ii)\ y_{(1,\alpha)(r,\nu_r)} = \sum_{\nu_q \in \Lambda_q} \sum_{\beta \in \Lambda_{n-1}} \sum_{\substack{\iota \in \Psi_{\alpha\beta}(y,z): \\ a_{q,\nu_q} \in \mathcal{P}_{\alpha\beta\iota}(y,z)}} z_{(1,\alpha)(r,\nu_r)(q,\nu_q)}$$

$$\forall \alpha \in \Lambda_1,\ \forall (r,q) \in R^2 : 1 < r < q,\ \forall\ \nu_r \in \Lambda_r$$

Proof. The lemma follows directly from the combination of constraints (9) and (10), and Theorems 20, 27, and 28, and Corollary 29. ■

Definition 31 (p.b.m. path "weight") *Let $(y,z) \in Q_L$. For $(\alpha,\beta) \in (\Lambda_1,\Lambda_{n-1})$ such that $y_{(1,\alpha)(n-1,\beta)} > 0$, and $k \in \Psi_{1,n-1}(y,z)$, we refer to the quantity*

$$\omega_{\alpha\beta k}(y,z) := \min_{\substack{(p,q)\in R^2;\ (\nu_p,\nu_q)\ \in\ (\Lambda_p,\Lambda_q): \\ 1<p<q;\ (a_{p,\nu_p},\ a_{q,\nu_q})\ \in\ P^2_{\alpha\beta k}(y,z)}} \left\{ z_{(1,\alpha)(p,\nu_p)(q,\nu_q)} \right\} \tag{50}$$

as the "weight" of (p.b.m. path) $\mathcal{P}_{\alpha\beta k}(y,z)$.

Remark 32 *It follows directly from Definitions 18 and 31 that for $(y,z) \in Q_L$, $\omega_{\alpha\beta\iota}(y,z) > 0$ $\forall$ $(\alpha,\beta) \in (\Lambda_1,\Lambda_{n-1}) : \Psi_{\alpha\beta}(y,z) \neq \varnothing$, $\forall$ $\iota \in P_{\alpha\beta\iota}(y,z)$.*

Theorem 33 (Path structure theorem 3) *Let $(y,z) \in Q_L$. The following hold true:*

i) $\forall$ $\alpha \in \Lambda_1$, $\forall$ $r \in R\backslash\{1\}$, $\forall$ $\rho \in \Lambda_r$,

$$y_{(1,\alpha)(r,\rho)} = \sum_{\beta\in\Lambda_{n-1}} \sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z): \\ a_{r,\rho}\ \in\ \mathcal{P}_{\alpha\beta\iota}(y,z)}} \omega_{\alpha\beta\iota}(y,z)$$

ii) $\forall (r,s) \in R^2 : 1 < r < s$, $\forall(\alpha,\rho,\sigma) \in (\Lambda_1,\Lambda_r,\Lambda_s)$,

$$z_{(1,\alpha)(r,\rho)(s,\sigma)} = \sum_{\beta\in\Lambda_{n-1}} \sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z): \\ (a_{r,\rho},\ a_{s\sigma})\ \in\ \mathcal{P}^2_{\alpha\beta\iota}(y,z)}} \omega_{\alpha\beta\iota}(y,z)$$

iii) $\forall (r,s) \in R^2 : 1 < r < s$, $\forall(\rho,\sigma) \in (\Lambda_r,\Lambda_s)$,

$$y_{(r,\rho)(s,\sigma)} = \sum_{\alpha\in\Lambda_1} \sum_{\beta\in\Lambda_{n-1}} \sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z): \\ (a_{r,\rho},\ a_{s,\sigma})\ \in\ \mathcal{P}^2_{\alpha\beta\iota}(y,z)}} \omega_{\alpha\beta\iota}(y,z)$$

Proof. $a)$ *Condition i*. First, note that from the combination of constraints (6), (7), (8), and (13); Remark 32; and Theorems (27), and (28), we must have:

$$\sum_{\alpha\in\Lambda_1} \sum_{\substack{\beta\in\Lambda_2: \\ i_{2,\beta}=j_{1,\alpha}}} \sum_{\substack{\delta\in\Lambda_3: \\ i_{3,\delta}=j_{2,\beta}}} z_{(1,\alpha)(2,\beta)(3,\delta)} = \sum_{\alpha\in\Lambda_1} \sum_{\varrho\in\Lambda_{n-1}} \sum_{\iota\in\Psi_{\alpha\varrho}(y,z)} \omega_{\alpha\varrho\iota}(y,z) = 1. \tag{51}$$

$a.1)$ From Definition 31, (51) $\Longrightarrow$

$$\sum_{\substack{\beta\in\Lambda_2: \\ i_{2,\beta}=j_{1,\alpha}}} \sum_{\substack{\delta\in\Lambda_3: \\ i_{3,\delta}=j_{2,\beta}}} z_{(1,\alpha)(2,\beta)(3,\delta)} = \sum_{\varrho\in\Lambda_{n-1}} \sum_{\iota\in\Psi_{\alpha\varrho}(y,z)} \omega_{\alpha\varrho\iota}(y,z) \quad \forall\alpha \in \Lambda_1 \tag{52}$$

Lemma 17 and relations (52) $\Longrightarrow$

$$\sum_{\rho\in\Lambda_r} \sum_{\sigma\in\Lambda_s} z_{(1,\alpha)(r,\rho)(s,\sigma)} = \sum_{\varrho\in\Lambda_{n-1}} \sum_{\iota\in\Psi_{\alpha\varrho}(y,z)} \omega_{\alpha\varrho\iota}(y,z) \quad \forall\alpha \in \Lambda_1, \forall(r,s) \in R^2 : 1 < r < s \tag{53}$$

Using constraints (9), (53) can be re-written as:

$$\sum_{\rho\in\Lambda_r} y_{(1,\alpha)(r,\rho)} = \sum_{\varrho\in\Lambda_{n-1}} \sum_{\iota\in\Psi_{\alpha\varrho}(y,z)} \omega_{\alpha\varrho\iota}(y,z) \quad \forall\alpha\in\Lambda_1, \forall r\in R\backslash\{1,n-1\} \tag{54}$$

Using Theorem 20, (54) can be written as:

$$\sum_{\rho\in\Lambda_r} y_{(1,\alpha)(r,\rho)} = \sum_{\rho\in\Lambda_r} \sum_{\varrho\in\Lambda_{n-1}} \sum_{\substack{\iota\in\Psi_{\alpha\varrho}(y,z):\\ a_{r,\rho}\,\in\,\mathcal{P}_{\alpha\varrho\iota}(y,z)}} \omega_{\alpha\varrho\iota}(y,z) \quad \forall\alpha\in\Lambda_1, \forall r\in R\backslash\{1,n-1\} \tag{55}$$

Re-arranging (55) gives:

$$\sum_{\rho\in\Lambda_r} \left(y_{(1,\alpha)(r,\rho)} - \sum_{\varrho\in\Lambda_{n-1}} \sum_{\substack{\iota\in\Psi_{\alpha\varrho}(y,z):\\ a_{r,\rho}\,\in\,\mathcal{P}_{\alpha\varrho\iota}(y,z)}} \omega_{\alpha\varrho\iota}(y,z) \right) = 0 \quad \forall\alpha\in\Lambda_1, \forall r\in R\backslash\{1,n-1\} \tag{56}$$

$a.2$) Combining Lemma 30.ii with Definition 31, we have that:

$$\begin{aligned} y_{(1,\alpha)(r,\rho)} &= \sum_{\nu_q\in\Lambda_q} \sum_{\varrho\in\Lambda_{n-1}} \sum_{\substack{\iota\in\Psi_{\alpha\varrho}(y,z):\\ a_{q,\nu_q}\,\in\,\mathcal{P}_{\alpha\beta\iota}(y,z)}} z_{(1,\alpha)(r,\rho)(q,\nu_q)} \\ &\geq \sum_{\nu_q\in\Lambda_q} \sum_{\varrho\in\Lambda_{n-1}} \sum_{\substack{\iota\in\Psi_{\alpha\varrho}(y,z):\\ (a_{r,\rho},\, a_{q,\nu_q})\,\in\,\mathcal{P}^2_{\alpha\beta\iota}(y,z)}} z_{(1,\alpha)(r,\rho)(q,\nu_q)} \\ &\geq \sum_{\varrho\in\Lambda_{n-1}} \sum_{\substack{\iota\in\Psi_{\alpha\varrho}(y,z):\\ a_{r,\rho}\,\in\,\mathcal{P}_{\alpha\varrho\iota}(y,z)}} \omega_{\alpha\varrho\iota}(y,z) \\ &\forall\ r\in R\backslash\{1,n-1\},\ \forall q\in R: q>r,\ \forall(\alpha,\rho)\in(\Lambda_1,\Lambda_r) \end{aligned} \tag{57}$$

Relations (56) and (57) $\Longrightarrow$

$$y_{(1,\alpha)(r,\rho)} = \sum_{\varrho\in\Lambda_{n-1}} \sum_{\substack{\iota\in\Psi_{\alpha\varrho}(y,z):\\ a_{r,\rho}\,\in\,\mathcal{P}_{\alpha\varrho\iota}(y,z)}} \omega_{\alpha\varrho\iota}(y,z) \quad \forall r\in R\backslash\{1,n-1\}, \forall(\alpha,\rho)\in(\Lambda_1,\Lambda_r) \tag{58}$$

$a.3$) Using constraints (10), (53) can be re-written as:

$$\sum_{\sigma\in\Lambda_s} y_{(1,\alpha)(s,\sigma)} = \sum_{\varrho\in\Lambda_{n-1}} \sum_{\iota\in\Psi_{\alpha\varrho}(y,z)} \omega_{\alpha\varrho\iota}(y,z) \quad \forall\alpha\in\Lambda_1, \forall s\in R: s>2 \tag{59}$$

Using Theorem 20, (59) $\Longrightarrow$

$$\sum_{\sigma\in\Lambda_s} y_{(1,\alpha)(s,\sigma)} = \sum_{\sigma\in\Lambda_s} \sum_{\varrho\in\Lambda_{n-1}} \sum_{\substack{\iota\in\Psi_{\alpha\varrho}(y,z):\\ a_{s,\sigma}\in\mathcal{P}_{\alpha\varrho\iota}(y,z)}} \omega_{\alpha\varrho\iota}(y,z) \quad \forall\alpha\in\Lambda_1, \forall s\in R: s>2 \tag{60}$$

Re-arranging (60) gives:

$$\sum_{\sigma\in\Lambda_s}\left(y_{(1,\alpha)(s,\sigma)}-\sum_{\varrho\in\Lambda_{n-1}}\sum_{\substack{\iota\in\Psi_{\alpha\varrho}(y,z):\\ a_{s,\sigma}\in\mathcal{P}_{\alpha\varrho\iota}(y,z)}}\omega_{\alpha\varrho\iota}(y,z)\right)=0\quad\forall\alpha\in\Lambda_1,\forall s\in R:s>2\tag{61}$$

$a.4$) Combining Lemma 30.i with Definition 31, we have that:

$$\begin{aligned}y_{(1,\alpha)(s,\sigma)}&=\sum_{\nu_p\in\Lambda_p}\sum_{\varrho\in\Lambda_{n-1}}\sum_{\substack{\iota\in\Psi_{\alpha\varrho}(y,z):\\ a_{p,\nu_p}\,\in\,\mathcal{P}_{\alpha\beta\iota}(y,z)}}z_{(1,\alpha)(p,\nu_p)(s,\sigma)}\\ &\geq\sum_{\nu_p\in\Lambda_p}\sum_{\varrho\in\Lambda_{n-1}}\sum_{\substack{\iota\in\Psi_{\alpha\varrho}(y,z):\\ (a_{p,\nu_p},\,a_{s,\nu_s})\,\in\,\mathcal{P}^2_{\alpha\beta\iota}(y,z)}}z_{(1,\alpha)(p,\nu_p)(s,\sigma)}\\ &\geq\sum_{\varrho\in\Lambda_{n-1}}\sum_{\substack{\iota\in\Psi_{\alpha\varrho}(y,z):\\ a_{s,\sigma}\,\in\,\mathcal{P}_{\alpha\varrho\iota}(y,z)}}\omega_{\alpha\varrho\iota}(y,z)\end{aligned}$$

$$\forall(p,s)\in R^2:1<p<s;\ s>2,\ \forall(\alpha,\sigma)\in(\Lambda_1,\Lambda_s)\tag{62}$$

$a.5$) Relations (61) and (62) $\Longrightarrow$

$$y_{(1,\alpha)(s,\sigma)}=\sum_{\varrho\in\Lambda_{n-1}}\sum_{\substack{\iota\in\Psi_{\alpha\varrho}(y,z):\\ a_{s,\sigma}\in\mathcal{P}_{\alpha\varrho\iota}(y,z)}}\omega_{\alpha\varrho\iota}(y,z)\quad\forall\alpha\in\Lambda_1,\forall s\in R:s>2\tag{63}$$

$a.6$) *Condition* i of the theorem follows from the combination of (58) and (63).

b) *Condition* ii.

$b.1$) Using Theorem 20 and Corollary 29.ii, (53) can be re-written as:

$$\sum_{\rho\in\Lambda_r}\sum_{\sigma\in\Lambda_s}\sum_{\beta\in\Lambda_{n-1}}\sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z):\\ (a_{r,\rho},\,a_{s,\sigma})\in P^2_{\alpha\beta\iota}(y,z)}}\omega_{\alpha\beta\iota}(y,z)=\sum_{\rho\in\Lambda_r}\sum_{\sigma\in\Lambda_s}z_{(1,\alpha)(r,\rho)(s,\sigma)}$$

$$\forall(r,s)\in R^2:1<r<s,\ \forall\alpha\in\Lambda_1\tag{64}$$

Re-arranging (64) gives:

$$\sum_{\rho\in\Lambda_r}\sum_{\sigma\in\Lambda_s}\left(z_{(1,\alpha)(r,\rho)(s,\sigma)}-\sum_{\beta\in\Lambda_{n-1}}\sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z):\\ (a_{r,\rho},\,a_{s,\sigma})\in P^2_{\alpha\beta\iota}(y,z)}}\omega_{\alpha\beta\iota}(y,z)\right)=0$$

$$\forall(r,s)\in R^2:1<r<s,\ \forall\alpha\in\Lambda_1\tag{65}$$

$b.2$) From Lemma 30.ii, we have:

$$y_{(1,\alpha)(s,\sigma)} = z_{(1,\alpha)(r,\rho)(s,\sigma)} + \sum_{\substack{\nu_r\in\Lambda_r:\\ \nu_r\neq\rho}} \sum_{\beta\in\Lambda_{n-1}} \sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z):\\ (a_{r,\nu_r},a_{s,\sigma})\in P^2_{\alpha\beta\iota}(y,z)}} z_{(1,\alpha)(r,\nu_r)(s,\sigma)}$$

$$\forall(r,s)\in R^2 : 1<r<s,\ \forall(\alpha,\rho,\sigma)\in(\Lambda_1,\Lambda_r,\Lambda_s). \tag{66}$$

$b.3$) From *Condition i*, we have:

$$y_{(1,\alpha)(s,\sigma)} = \sum_{\beta\in\Lambda_{n-1}} \sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z):\\ (a_{r,\rho},\ a_{s,\sigma})\in P^2_{\alpha\beta\iota}(y,z)}} \omega_{\alpha\beta\iota}(y,z) \ + \sum_{\substack{\nu_r\in\Lambda_r:\\ \nu_r\neq\rho}} \sum_{\beta\in\Lambda_{n-1}} \sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z):\\ (a_{r,\nu_r},a_{s,\sigma})\in P^2_{\alpha\beta\iota}(y,z)}} \omega_{\alpha\beta\iota}(y,z)$$

$$\forall(r,s)\in R^2 : 1<r<s,\ \forall(\alpha,\rho,\sigma)\in(\Lambda_1,\Lambda_r,\Lambda_s) \tag{67}$$

$b.4$) Definition 31 $\Longrightarrow$:

$$\sum_{\substack{\nu_r\in\Lambda_r:\\ \nu_r\neq\rho}} \sum_{\beta\in\Lambda_{n-1}} \sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z):\\ (a_{r,\nu_r},a_{s,\sigma})\in P^2_{\alpha\beta\iota}(y,z)}} z_{(1,\alpha)(r,\nu_r)(s,\sigma)} \geq \sum_{\substack{\nu_r\in\Lambda_r:\\ \nu_r\neq\rho}} \sum_{\beta\in\Lambda_{n-1}} \sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z):\\ (a_{r,\nu_r},a_{s,\sigma})\in P^2_{\alpha\beta\iota}(y,z)}} \omega_{\alpha\beta\iota}(y,z)$$

$$\forall(r,s)\in R^2 : 1<r<s,\ \forall(\alpha,\rho,\sigma)\in(\Lambda_1,\Lambda_r,\Lambda_s) \tag{68}$$

$b.5$) Relations (66)-(68) $\Longrightarrow$

$$z_{(1,\alpha)(r,\rho)(s,\sigma)} \leq \sum_{\beta\in\Lambda_{n-1}} \sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z):\\ (a_{r,\rho},a_{s,\sigma})\in P^2_{\alpha\beta\iota}(y,z)}} \omega_{\alpha\beta\iota}(y,z)$$

$$\forall(r,s)\in R^2 : 1<r<s,\ \forall(\alpha,\rho,\sigma)\in(\Lambda_1,\Lambda_r,\Lambda_s) \tag{69}$$

$b.6$) Combining (65) and (69), we must have:

$$z_{(1,\alpha)(r,\rho)(s,\nu_s)} = \sum_{\beta\in\Lambda_{n-1}} \sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z):\\ (a_{r,\rho},a_{s,\sigma})\in P^2_{\alpha\beta\iota}(y,z)}} \omega_{\alpha\beta\iota}(y,z)$$

$$\forall(r,s)\in R^2 : 1<r<s,\ \forall(\alpha,\rho,\sigma)\in(\Lambda_1,\Lambda_r,\Lambda_s) \tag{70}$$

c) *Condition iii*. From the combination of constraints (12) and *Condition ii*, we have:

$$y_{(r,\rho)(s,\sigma)} = \sum_{\alpha\in\Lambda_1} z_{(1,\alpha)(r,\rho)(s,\sigma)} = \sum_{\alpha\in\Lambda_1} \sum_{\beta\in\Lambda_{n-1}} \sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z):\\ (a_{r,\rho},\ a_{s,\sigma})\in P^2_{\alpha\beta\iota}(y,z)}} \omega_{\alpha\beta\iota}(y,z)$$

$$\forall(r,s)\in R^2 : 1<r<s,\ \forall(\rho,\sigma)\in(\Lambda_r,\Lambda_s). \tag{71}$$

■

Corollary 34 *$(y,\, z) \in Q_L \iff (y,z)$ corresponds to a convex combination of perfect matchings of facilities and sites with coefficients equal to the weights of the corresponding p.b.m.paths in (y,z).*

Theorem 35 *The following holds true: $Conv(Q_L) = Conv(Q_I)$.*

Proof. The theorem follows directly from the combination of Theorem 26, Theorem 27, and Corollary 34. ■

Corollary 36 *The following mappings are bijective:*

1. $\mathcal{B}_1 : Ext(Q_L) \longmapsto \Omega$;
2. $\mathcal{B}_2 : Conv(Q_L) \longmapsto Conv(W_L)$
3. $\mathcal{B}_3 : Ext(Q_L) \longmapsto Ext(W_L)$
4. $\mathcal{B}_4 : Conv(Q_L) \longmapsto Conv(W_I)$
5. $\mathcal{B}_5 : Ext(Q_L) \longmapsto Ext(Conv(W_I))$

2.2 Model Objective

Definition 37 (Objective function costs) *Let $(y, z) \in Q_L$. $\forall (i, j, u, v, k, t) \in F^6$, $\forall (r, s) \in R^2$: $1 < r < s$, the "cost" associated with $z_{u1,virjkst}$ is defined as:*

$$c_{u1,virjkst} := \begin{cases} o_{u,1} + o_{v,2} + h_{u,1,v,2} + h_{u,1,j,r+1} + h_{u,1,t,s+1} + h_{v,2,j,r+1} + h_{v,2,t,s+1} \\ \text{if } \ r = 2; s = 3; i = v; k = j \\ \\ h_{u,1,t,s+1} + h_{v,2,t,s+1} \quad \text{ if } r = 2; s \geq 4; i = v \\ \\ o_{ir} + h_{irj,r+1} + h_{irt,r+2} \quad \text{ if } \ 3 \leq r \leq n-3; s = r+1; k = j \\ \\ o_{ir} + o_{j,r+1} + o_{t,r+2} + h_{irj,r+1} + h_{irt,r+2} + h_{j,r+1,t,r+2} \\ \text{if } \ r = n-2; s = n-1; k = j \\ \\ h_{irt,s+1} \quad \text{ if } \ 3 \leq r \leq n-3; s \geq r+2 \\ \\ 0 \ \text{ Otherwise} \end{cases}$$

Theorem 38 *Let:*

$$\vartheta(y, z) := c^T \cdot z \ + \ \mathbf{0}^T \cdot y = \sum_{u \in F} \sum_{v \in F} \sum_{i \in F} \sum_{\substack{r \in R: \\ r > 1}} \sum_{j \in F} \sum_{k \in F} \sum_{\substack{s \in R: \\ s > r}} \sum_{t \in F} c_{u,1,virjkst} z_{u,1,virjkst}. \tag{72}$$

Then, $\vartheta(y, z)$ accurately accounts the cost of the perfect matching of the facilities and sites that is associated with (y, z) for all $(y, z) \in Ext(Q_L)$.

Proof. From Theorem 35,

$$(y, z) \in Ext(Q_L) \Longrightarrow (y, z) \in Q_I \tag{73}$$

Now, using Theorems 9, it can be verified directly that for $(y, z) \in Q_I$,

$\vartheta(y, z) = \sum\limits_{r=1}^{n-1} \sum\limits_{s=r+1}^{n} h_{i_r,r,i_s,s} + \sum\limits_{r=1}^{n} o_{i_r,r}$, where $i_r \in \mathcal{M}(y, z) \quad \forall \ r \in T$. ■

2.3 Overall Model

Our overall linear programming model is as follows:

Problem 39 (*Problem LP*)

$$min\ \{\vartheta(y,z) : (y,z) \in Q_L\}$$

Theorem 40 *The following statements are true of basic feasible solutions (BFS) of Problem LP and perfect matchings of the facilities and sites:*

1. Every BFS of *Problem LP* corresponds to a perfect matching of the facilities and sites;

2. Every perfect matching of the facilities and sites corresponds to a BFS of *Problem LP*;

3. The mapping of BFS's of *Problem LP* onto the set of perfect matchings of the facilities and sites is surjective.

Proof. Statements (40.1) and (40.2) follow directly from Theorem 35, and the correspondence between BFS's of LP models and extreme points of polyhedra (see [2, pp. 92-101]). Statement (40.3) follows from the primal degeneracy of *Problem LP* (see [11, p. 32]). ■

Corollary 41 *Problem LP and Problem QAP are equivalent; that is, Problem LP correctly solves the QAP.*

3 Numerical Implementation

As indicated in section 1 and the abstract of this paper, we conducted a small numerical experimentation aimed at getting some rough idea about the computational performance of the model. In doing this, we implemented a streamlined version of the model (i.e., *Problem LP*) in which constraints (12)-(14) were handled implicitly, and the upper bounds on the variables were omitted. We solved one set of five problems with 7 facilities, and one set of five problems with 8 facilities. Four (of the five) problems in each of the two sets were randomly generated, and one problem had all parameters (i.e., each inter-site distance, each inter-facility flow, and each operating cost).equal to zero. In the randomly-generated problems, the inter-facility flows and the inter-site distances were assumed to be uniform random numbers between 0 and 300, and between 1 and 50, respectively. The facility operating costs were assumed to be zero in two of the randomly-generated problems in each set, and assumed to be random deviates on [0, 1000] for the remainder two problems in the set. Furthermore, one of the two problems with operating costs equal to zero had asymmetric inter-facilities flows and asymmetric inter-sites distances, while the other two problems had symmetric inter-facility flows and symmetric inter-site distances. Similarly, one of the two problems with operating costs greater than zero had symmetric inter-facilities flows and symmetric inter-sites distances, while the other two problems had asymmetric inter-facility flows and asymmetric inter-site distances. We solved each of the test problems using the simplex procedure on the primal and dual forms, respectively, and using the primal-dual interior-point (”barrier”) method. The linear programming (LP) solver we used is the ”Clp” set of routines of the *COIN-OR* open source library ([4]).

The characteristics of the test problems and the computational results are summarized in Table 1. The average computational time for the 7-site problems was 24.76 seconds, 59.44 seconds, and 305.50 seconds of Sony VAIO notebook computer (1.83 GHz Intel Core 2 Duo processor) for the simplex procedure for primal form, the simplex procedure for the dual form, and the barrier method, respectively. The corresponding numbers for the 8-site problems, were 8,988.18 seconds, 3,064.86 seconds, and 3,765.05 seconds, respectively.

In general, it appeared that the computational performance of the model is not affected by the symmetry/asymmetry of the inter-site distances, nor by the symmetry/asymmetry of the inter-facilities flows, nor by the presence/absence of fixed facilties operating costs. Both the simplex method using the dual form, and the barrier method appeared to perform better than the simplex method using the primal form when the number of sites is increased. However, it also appeared that this could have been due to numerical stability problems. Overall, we believe the simplex procedure using the primal form may hold the greatest promise with respect to further developments aimed at solving larger-sized problems, based on the fact that it tended to examine a small number of the perfect matchings of the facilities and sites in general.

Problem Details				Simplex: Primal		Simplex: Dual		Barrier Method	
$n^{(1)}$	$F^{(2)}$	$D^{(3)}$	$O^{(4)}$	# of iter.	CPU sec.	# of iter.	CPU sec.	# of iter.	CPU sec.
7	A	A	Y	19,167	22.75	16,434	53.31	17,048	295.53
7	A	A	N	24,610	36.062	21,189	84.06	22,326	355.61
7	S	S	Y	20,365	27.14	16,504	50.84	1,224	272.91
7	S	S	N	21,990	34.52	19,467	71.20	20,038	441.92
7	N	N	N	10,265	3.31	10,610	37.77	5,284	161.53
Average				19,274.4	24.76	16,840.8	59.44	13,184.0	305.50
8	A	A	Y	357,848	10,837.66	115,794	2,763.28	107,150	3,589.19
8	A	A	N	368,682	11,023.23	118,512	3,036.61	110,554	3,760.24
8	S	S	Y	379,912	11,777.20	132,518	3,733.08	124,685	4,469.31
8	S	S	N	376,547	11,266.19	131,622	3,711.48	124,897	4,977.52
8	N	N	N	32,831	36.64	67,559	2,079.84	69,498	2,028.98
Average				303,164.0	8,988.18	113,201.0	3,064.86	107,356.8	3,765.05

1: Number of sites
2: Flow type: "A"=asymmetric; "S"=symmetric; "N"=All flows are zero
3: Distance type: "A"=asymmetric; "S"=symmetric; "N"=All distances are zero
4. Operating Costs: "Y"=Not all equal to zero; "N"=All equal to zero

Table 1: Summary of the computational experimentation

4 Conclusions

We have developed a linear programming model of the QAP. From a theoretical perspective, the proposed model provides a new affirmative resolution to the important issue of the equality of computational complexity classes P and NP. Although our proposed linear programming model has a polynomially–bounded size ($O(n^8)$ variables $\times$ $O(n^7)$ constraints), that size is still prohibitive with respect to practice. Hence, we believe developments aimed at solving larger-sized problems could be the subject of fruitful future research.

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